1.(6pts)Let D be the region in the plane bounded by the circle $x^2 + y^2 = 4$ with density $\rho(x, y) = x^2 + y^2$. Which of the following statements is true:

(a) The center of mass of D is the origin.

(b) The center of mass of D is $(1, 1)$

(c) The center of mass of D is $(0.5, -0.5)$

(d) The center of mass of D is $(-0.5, 0.5)$

(e) The center of mass of D is $(2, -2)$

Solution. The region D and the density $\rho(x, y)$ are symmetric about the x-axis, hence the center of mass is located in the x-axis. Similarly, D and $\rho(x, y)$ are symmetric about the y-axis, so the center of mass is located in the y-axis. Therefore, the center of mass of D is the origin.

2.(6pts) Calculate the volume enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$.

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{1}{2}$ (d) 2π (e) 1

Solution. Using cylindrical co-ordinates, the paraboloid is $z = r^2$. The surfaces intersect when $r^2 = 1$, ie, $r = 1$. It follows that the volume is

$$\int_0^{2\pi} \int_0^1 \int_{r^2}^1 r \, dz \, dr \, d\phi = 2\pi \int_0^1 \left(r - r^3\right) dr = 2\pi \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^1 = \frac{\pi}{2}$$

3.(6pts) Let S be the solid inside both the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 1$. Write the iterated integral $\iiint\limits_S z \, dV$ in spherical coordinates.

(a) $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$

(b) $\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$

(c) $\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$

(d) $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^2 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$

(e) $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho \cos\phi \, d\rho \, d\phi \, d\theta$

Solution. The cone $z = \sqrt{x^2+y^2}$ is the same as $\phi = \frac{\pi}{4}$ in spherical coordinates and the sphere $x^2+y^2+z^2 = 1$ is $\rho = 1$. The limits of integration are therefore $0 \le \rho \le 1$, $0 \le \phi \le \frac{\pi}{4}$ and $0 \le \theta \le 2\pi$. The iterated integral is therefore

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

4.(6pts) Let a thin wire in space be in the shape of the helix C given by $\mathbf{r}(t) = \left\langle \cos 3t, \sin 3t, 3\sqrt{3}\, t \right\rangle$, $0 \le t \le 2\pi$. Let the linear density at (x, y, z) be given by $\mu(x, y, z) = \frac{z}{x^2+y^2}$. Compute the mass of the wire.

(a) $36\sqrt{3}\,\pi^2$ (b) $216\sqrt{3}\,\pi^2$ (c) $36\sqrt{3}\,\pi$ (d) $6\sqrt{3}\,\pi^2$ (e) 12π

Solution. Recall

$$\int_C \mu(x,y,z)\, ds = \int_0^{2\pi} f\big(\mathbf{r}(t)\big)\, |\mathbf{r}'(t)|\, dt$$

First, we find $\mathbf{r}'(t)$:

$$\mathbf{r}'(t) = \left\langle -3\sin 3t, 3\cos 3t, 3\sqrt{3} \right\rangle$$

then

$$|\mathbf{r}'(t)| = \sqrt{(-3\sin 3t)^2 + (3\cos 3t)^2 + (3\sqrt{3})^2} = \sqrt{9+27} = \sqrt{36} = 6$$

So, the mass is

$$\begin{aligned}
\int_C \mu(x,y,z)\, ds &= \int_0^{2\pi} \mu(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt \\
&= \int_0^{2\pi} \frac{3\sqrt{3}\, t}{(\cos 3t)^2 + (\sin 3t)^2}(6) dt \\
&= \int_0^{2\pi} 18\sqrt{3}\, t\, dt \\
&= 9\sqrt{3}\, t^2 \Big|_0^{2\pi} = 36\sqrt{3}\pi^2
\end{aligned}$$

5.(6pts) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = -2xy\mathbf{i} + 4y\mathbf{j} + \mathbf{k}$ and $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}, 0 \le t \le 2$.

(a) 24 (b) 12 (c) 48 (d) 18 (e) 32

Solution. Since $\mathbf{F}(\mathbf{r}(t)) =< -2t^3, 4t^2, 1 >$ and $\mathbf{r}' =< 1, 2t, 0 >$. We get that $\mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(\mathbf{t}) = 6t^3$. Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 6t^3 dt = 24$.

6.(6pts) Let $\mathbf{F} = \nabla f$ where $f(x, y) = x^2\cos(y) + e^{\sin(y)}$. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the semicircle $x^2+y^2 = 1, x \ge 0$ and the orientation is counterclockwise.

(a) $e^{\sin(1)} - e^{\sin(-1)}$ (b) $e^{\sin(-1)} - e^{\sin(1)}$ (c) -2

(d) 2 (e) $\cos(1) + \cos(-1)$

Solution. Using the Fundamental Theorem of Line Integrals, we see that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0,1) - f(0,-1) = (0 + e^{\sin(1)}) - (0 + e^{\sin(-1)}) = e^{\sin(1)} - e^{\sin(-1)}$$

7.(6pts) Rewrite the iterated integral

$$\int_0^3 \int_0^{2-\frac{2z}{3}} \int_0^{6-2z-3y} \frac{e^z}{(z+2)(6-z)}\, dx\, dy\, dz$$

as an iterated integral $dy\, dx\, dz$. In other words, switch $dx\, dy$ to $dy\, dx$.

(a) $\displaystyle\int_0^3 \int_0^{6-2z} \int_0^{\frac{6-2z-x}{3}} \frac{e^z}{(z+2)(6-z)}\, dy\, dx\, dz$

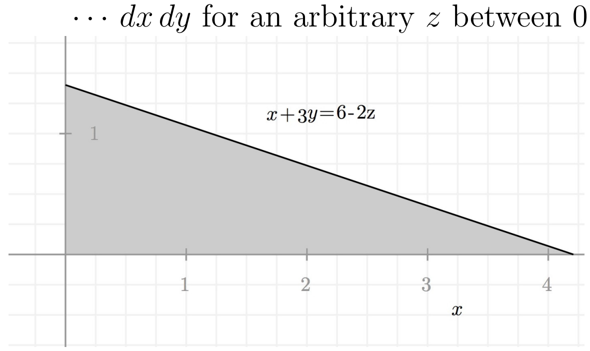
(b) $\displaystyle\int_0^3 \int_0^{2-\frac{2z}{3}} \int_0^{6-2z-3x} \frac{e^z}{(z+2)(6-z)}\, dy\, dx\, dz$

(c) $\displaystyle\int_0^3 \int_0^{6-2z} \int_0^{6-2z-3x} \frac{e^z}{(z+2)(6-z)}\, dy\, dx\, dz$

(d) $\displaystyle\int_0^3 \int_0^{6-2z-3x} \int_0^{6-2z} \frac{e^z}{(z+2)(6-z)}\, dy\, dx\, dz$

(e) $\displaystyle\int_0^3 \int_0^{6-2y-3x} \int_0^{6-2x} \frac{e^z}{(z+2)(6-z)}\, dy\, dx\, dz$

Solution. The outer integral is the same in both cases so start by drawing the plane region determined by $\displaystyle\int_0^{2-\frac{2z}{3}} \int_0^{6-2z-3y} \cdots\, dx\, dy$ for an arbitrary z between 0 and 1.



The integral in the other order has the outside integral running from $x = 0$ to $x = 6 - 2z$. In the inner integral, y starts at 0 and runs to $\dfrac{6-2z-x}{3}$

8.(10pts) Let S be the "ice-cream cone" bounded below by $z = \sqrt{3(x^2+y^2)}$ and above by $x^2 + y^2 + z^2 = 4$. Use spherical coordinates to express the volume of S as an integral. **Do not evaluate the integral.**

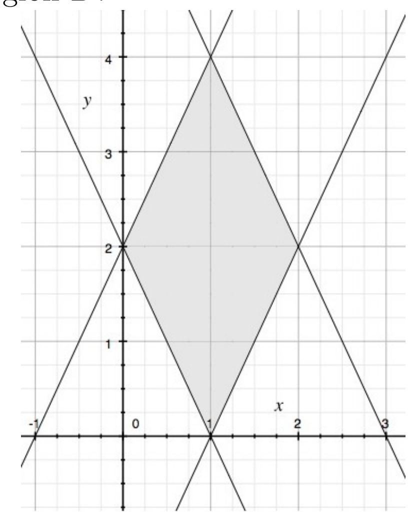
Solution. The sphere $x^2 + y^2 + z^2 = 4$ is $\rho = 2$ and the cone $z = \sqrt{3(x^2+y^2)}$ transforms as $\phi = \dfrac{\pi}{6}$. The volume is then given by

$$\iiint_S dV = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

9.(10pts) Consider the double integral $\iint\limits_D y\,dA$ where D is the region bounded by $y+2x=2$, $y+2x=6$, $y-2x=-2$, and $y-2x=2$. Let T^{-1} be the transformation defined by $u=y+2x$ and $w=y-2x$. Describe the region R such that $T(R)=D$ and write $\iint\limits_D y\,dA$ as a double integral over R. Note $x=\dfrac{1}{4}(u-w)$ and $y=\dfrac{1}{2}(u+w)$.

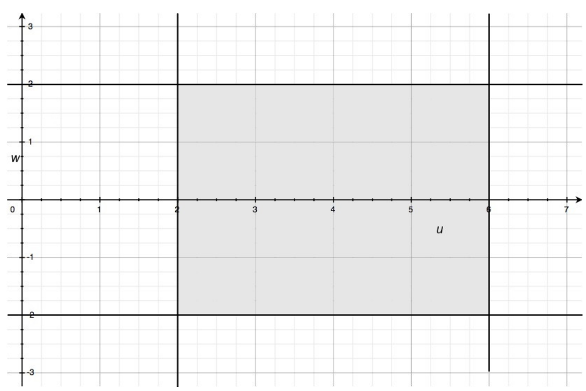
Do not evaluate the integral.

Solution. First let draw the region D.



In terms of u and w, $y + 2x = 2$ is equivalent to $u = 2$.
In terms of u and w, $y + 2x = 6$ is equivalent to $u = 6$.
In terms of u and w, $y - 2x = 2$ is equivalent to $w = 2$.
In terms of u and w, $y - 2x = 2$ is equivalent to $w = -2$.

The region R is the rectangle with corners (u, w) equal to $(2, 2)$, $(6, 2)$, $(2, -2)$, $(6, -2)$. Here is a picture.



Next So $\dfrac{\partial(x,y)}{\partial(u,w)} = \det \begin{vmatrix} \dfrac{\partial \frac{1}{4}(u-w)}{\partial u} & \dfrac{\partial \frac{1}{4}(u-w)}{\partial w} \\ \dfrac{\partial \frac{1}{2}(u+w)}{\partial u} & \dfrac{\partial \frac{1}{2}(u+w)}{\partial w} \end{vmatrix} = \det \begin{vmatrix} \dfrac{1}{4} & -\dfrac{1}{4} \\ \dfrac{1}{2} & \dfrac{1}{2} \end{vmatrix} = \dfrac{1}{8} - \left(-\dfrac{1}{8}\right) = \dfrac{1}{4}.$

Then

$$\iint_D y\, dA = \iint_R \left(\frac{u+w}{2} \cdot \frac{1}{4}\right) dA$$

10.(10pts) Find a potential function for the field $\left\langle 2xe^{x^2y} + 2x^3ye^{x^2y} + 2xe^{x^2}, x^4e^{x^2y} + 2y \right\rangle$.

Solution.

$$\frac{\partial p}{\partial y} = x^4 e^{x^2 y} + 2y$$

so

$$p = x^2 e^{x^2 y} + y^2 + h(x)$$

$$\frac{\partial p}{\partial x} = \frac{\partial x^2 e^{x^2 y} + y^2 + h(x)}{\partial x} = 2xe^{x^2 y} + x^2(2xy)e^{x^2 y} + h'(x) = 2xe^{x^2 y} + 2x^3 y e^{x^2 y} + 2xe^{x^2}$$

so

$$h'(x) = 2xe^{x^2}$$

$$p = x^2 e^{x^2 y} + y^2 + e^{x^2}$$

OR

Start with

$$\frac{\partial p}{\partial x} = 2xe^{x^2 y} + 2x^3 y e^{x^2 y} + 2xe^{x^2}$$

This is harder to integrate but it can be done. An anti-derivative of the first term is $\frac{e^{x^2 y}}{y}$. An anti-derivative of the third term is e^{x^2}. To find an anti-derivative for the second term, use the substitution $t = x^2$, so $dt = 2x\,dx$ and $\int 2x^3 y e^{x^2 y}\,dx = \int yte^{yt}dt$. This is a parts: $u = t$ $dv = e^{yt}dt$ so $du = dt$ and $v = \frac{e^{yt}}{y}$.

Hence $\int 2x^3 y e^{x^2 y}\,dx = \int yte^{yt}dt = yt\frac{e^{yt}}{y} - y\int \frac{e^{yt}}{y}\,dt = te^{yt} - \frac{e^{yt}}{y} = x^2 e^{x^2 y} - \frac{e^{x^2 y}}{y}$.

Therefore

$$p = \frac{e^{x^2 y}}{y} + x^2 e^{x^2 y} - \frac{e^{x^2 y}}{y} + e^{x^2} + h(y) = x^2 e^{x^2 y} + e^{x^2} + h(y)$$

Then

$$\frac{\partial p}{\partial y} = x^4 e^{x^2 y} + h'(y) = x^4 e^{x^2 y} + 2y$$

so $h'(y) = 2y$ and $h(y) = y^2$ so

$$p = x^2 e^{x^2 y} + e^{x^2} + y^2$$

is a potential function.

11.(10pts) Use Green's Theorem to write down a double integral over the shaded region in Figure 12 which is equal to the line integral

$$\int_C \big(y^2 + (\sin x)e^x - 2x\big)\,dx + \big(x^2 + (y\cos y)e^{y^2} + 2 - 3x\big)\,dy$$

along the boundary of the region.

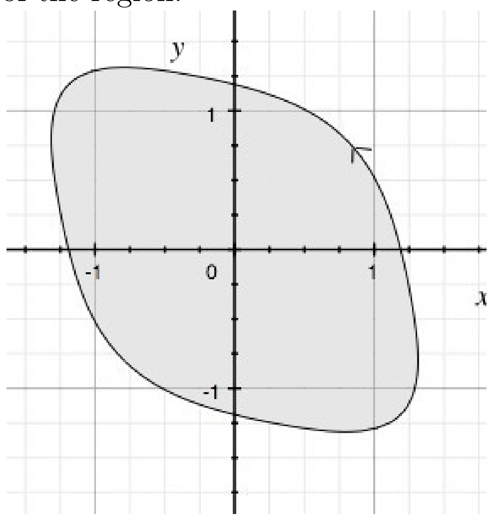


Figure 12.

Solution. Let D be the shaded region. $M = y^2 + (\sin x)e^x - 2x$ and $\dfrac{\partial M}{\partial y} = 2y$. $N = x^2 + (y\cos y)e^{y^2} + 2 - 3x$ and $\dfrac{\partial N}{\partial x} = 2x - 3$. Hence $\displaystyle\int_{\partial D} \big(y^2 + (\sin x)e^x - 2x\big)dx + \big(x^2 + (y\cos y)e^{y^2} + 2 - 3x\big)dy = \iint_D (2x - 3 + 2y)\,dA.$

Now $\displaystyle\int_{\partial D} \big(y^2 + (\sin x)e^x - 2x\big)dx + \big(x^2 + (y\cos y)e^{y^2} + 2 - 3x\big)dy$ is the line integral around the boundary curve oriented so that the region is on your left. This is the given orientation on C so $\displaystyle\int_C \big(y^2 + (\sin x)e^x - 2x\big)dx + \big(x^2 + (y\cos y)e^{y^2} + 2 - 3x\big)dy = \iint_D (2x - 3 + 2y)\,dA.$